

Nonlinear autoregressive
time series models in R
using **tsDyn** version 0.7

1 Introduction

tsDyn is an R package for the estimation of a number of nonlinear time series models. The package is at an early stage, and may presumably change significantly in the near future. However, it is quite usable in the current version.

Each function in the package has at least a minimal help page, with one or more working examples and detailed explanation of function arguments and returned values. In this document we try to give an overall guided tour of package contents, with some additional notes which are generally difficult to put in the context of a manual.

This guide is divided into 3 main sections:

- Explorative analysis tools
- Nonlinear autoregressive models
- A case study

2 Explorative analysis

2.1 Bivariate and trivariate relations

A first explorative analysis should include inspecting the distribution of (x_t, x_{t-l}) and that of $(x_t, x_{t-l_1}, x_{t-l_2})$ for some lags l, l_1, l_2 . This can be done easily in R in a variety of ways. The **tsDyn** package provide functions **autopairs** and **autotriples** for this purpose.

The **autopairs** function displays, in essence, a scatterplot of time series x_t versus x_{t-lag} . The main arguments to the function are the time series and the desired lag. The scatterplot may be also processed to produce bivariate kernel density estimations, as well as nonparametric kernel autoregression estimations. The type of output is governed by the argument **type**. Possible values, along with their meanings, are:

lines	directed lines
points	simple scatterplot
levels	iso-density levels
persp	density perspective plot
image	density image map
regression	kernel autoregression line superposed to scatterplot

For kernel density and regression estimation, you can specify also the kernel window **h**. A typical call to that function can be:

R code

```
autopairs(x, lag = , type = , h = )
```

All arguments (except the time series **x**) have default values.

By default, if running in an interactive environment, the function displays a simple experimental cross-platform GUI, where you can change function parameters and watch interactively how the plot changes.

Similar to **autopairs**, there is the **autotriples** function. This shows x_t versus (x_{t-lag1}, x_{t-lag2}) , so that the user has to specify time series x and lags **lag1** and **lag2**. The scatterplot can be processed to produce kernel regression estimates. Plotting possibilities are:

```
levels  iso-values lines
persp   perspective plot
image   image map
lines   directed lines
points  simple scatterplot
```

2.2 Linearity

An interesting tool for inspecting possible nonlinearities in the time series is the *locally linear autoregressive fit* plot, proposed by Casdagli [1]. Suppose you think that the dynamical system underlying your time series is best reconstructed with embedding dimension m and time delay d . Then the locally linear autoregressive fit plot displays the relative error made by forecasting time series values with linear models of the form:

$$x_{t+s} = \phi_0 + \phi_1 x_t + \dots + \phi_m x_{t-(m-1)d}$$

estimated on points in the sphere of radius ϵ around \mathbf{x}_t^m for a range of values of ϵ . A minimum attained at relatively small values of ϵ may indicate that a global linear model would be inappropriate for the approximation of the time series dynamics.

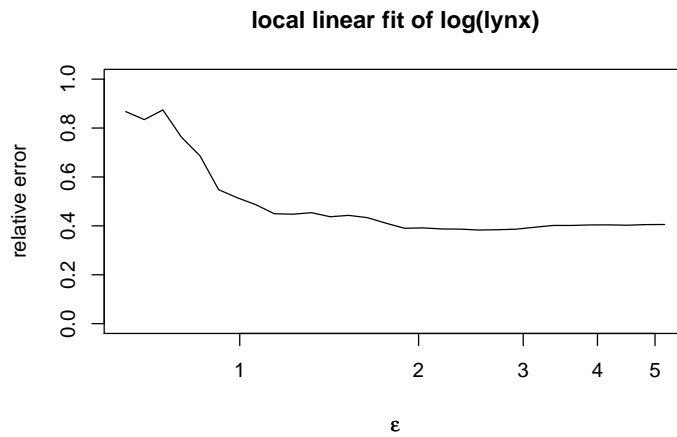
For this analysis **tsDyn** proposes the function **llar** which accepts, among others, the following arguments:

```
x    time series
m, d, steps  embedding parameters (see the above model formulation)
```

The function returns a ‘**llar**’ object, which can be plotted with the generic **plot** method. So, a typical usage would be:

R code

```
obj <- llar(log(lynx), m = 3)
plot(obj)
```



However, the `obj` object can be explicitly converted in an ordinary `data.frame`:

R code

```
obj <- data.frame(obj)
```

with variables:

R code

```
names(obj)
```

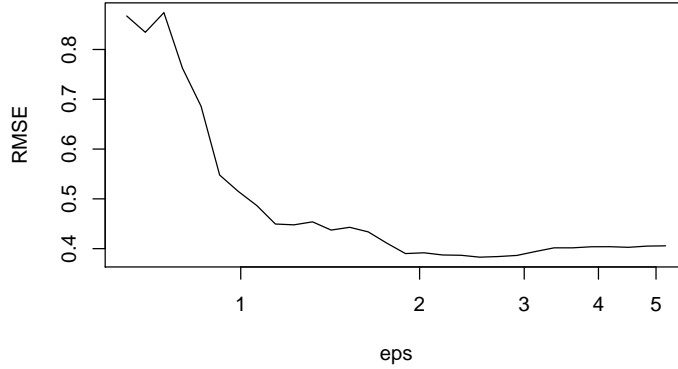
output

```
[1] "RMSE"    "eps"     "frac"    "avfound"
```

where ‘RMSE’ stands for Relative Mean Square Error, and `eps` is enough self-explaining. You can explore this object with usual R commands dedicated to `data.frames`, such as:

R code

```
plot(RMSE ~ eps, data = obj, type = "l", log = "x")
```



2.3 Tests (experimental)

`tsDyn` implements conditional mutual independence and linearity tests as described in Manzan [5]. Function implementations are rather basic, and little tested. Use them carefully!

The `delta.test` function performs a bootstrap test of independence of x_t versus x_{t-md} conditional on intermediate observations $\{x_{t-d}, \dots, x_{t-(m-1)d}\}$. The test statistic, available with the function `delta`, is based on the sample correlation integral, and calls internally the `d2` function provided by the `tseriesChaos` package. Among others things, the test requires the specification of a neighborhood window ϵ .

Function arguments are the time series `x`, a vector of embedding dimensions `m`, time delay `d`, a vector of neighborhood windows `eps`, the number of bootstrap replications `B`. However, default values are available for `m`, `d`, `eps` and `B`, so that a typical call can be:

R code

```
delta.test(x)
```

The return value is a matrix of p-values, labelled with their associated embedding dimensions and neighborhood windows (normally multiple values are tried simultaneously).

The `delta.lin.test` function performs a bootstrap test of linear dependence of x_t versus x_{t-md} conditional on intermediate observations $\{x_{t-d}, \dots, x_{t-(m-1)d}\}$. The test statistic is available with the function `delta.lin`. The function arguments and returned values are the same as those of `delta.test`.

3 Nonlinear autoregressive time series models

Consider the discrete-time univariate stochastic process $\{X_t\}_{t \in T}$. Suppose X_t is generated by the map:

$$X_{t+s} = f(X_t, X_{t-d}, \dots, X_{t-(m-1)d}; \theta) + \epsilon_{t+s} \quad (1)$$

with $\{\epsilon_t\}_{t \in T}$ white noise, ϵ_{t+s} independent w.r.t. X_{t+s} , and with f a generic function from \mathbf{R}^m to \mathbf{R} . This class of models is frequently referenced in the literature with the acronym NLAR(m), which stands for *NonLinear AutoRegressive* of order m .

In (1), we have implicitly defined the *embedding dimension* m , the *time delay* d and the *forecasting steps* s . The vector θ indicates a generic vector of parameters governing the shape of f , which we would estimate on the basis of some empirical evidence (i.e., an observed time series $\{x_1, x_2, \dots, x_N\}$).

In `tsDyn` some specific NLAR models are implemented. For a list of currently available models, type:

```

_____ R code _____
availableModels()
_____
_____ output _____
[1] "linear" "nnetTs" "setar" "lstar" "star" "aar"
_____

```

Each model can be estimated using a function which takes the name of the model as indicated by `availableModels`. I.e., use `linear` for fitting a linear model.

All those functions returns an object of base class `nlar`, from which informations can be extracted using some common methods. Among others:

```

print(obj) #prints basic infos on fitted model and estimated parameters
summary(obj) #if possible, shows more detailed infos and diagnostics on estimated model
plot(obj) #shows common diagnostic plots

```

Another method that can be useful for inspecting the estimated model properties is the `predict` method:

```

_____ R code _____
x.new <- predict(obj, n.ahead = )
_____

```

This function attempts to extend of `n.ahead` observations of the original time series used for estimating the model encapsulated in `obj` using the so called

skeleton of the fitted model. Assuming that from (1) we estimated f as $\hat{f} = f(\cdot; \hat{\theta})$, using the time series $\mathbf{x} = \{x_1, x_2, \dots, x_N\}$, we have:

$$\begin{aligned}\hat{x}_{N+1} &= \hat{f}(x_{N-s}, x_{N-s-d}, \dots, x_{N-s-(m-1)d}) \\ \hat{x}_{N+2} &= \hat{f}(x_{N-s+1}, x_{N-s+1-d}, \dots, x_{N-s+1-(m-1)d}) \\ &\dots \\ \hat{x}_{N+S} &= \hat{f}(x_{N-s+(S-1)}, x_{N-s+(S-1)-d}, \dots, x_{N-s+(S-1)-(m-1)d})\end{aligned}$$

A detailed description of some actually implemented models follows.

3.1 Linear models

$$X_{t+s} = \phi + \phi_0 X_t + \phi_1 X_{t-d} + \dots + \phi_m X_{t-(m-1)d} + \epsilon_{t+s} \quad (2)$$

It's a classical AR(m) model, and its specification doesn't require additional hyper-parameters. Estimation is done via CLS (Conditional Least Squares). The summary command returns asymptotics standard errors for the estimated coefficients, based on the normality assumption of residuals.

Note that in R there are plenty of functions for AR (and, more generally, ARMA) models estimation, with different estimation methods, such as ML, CLS, Yule-Walker, ... If you really need to fit linear models, use these methods directly.

3.2 SETAR models

$$X_{t+s} = \begin{cases} \phi_1 + \phi_{10} X_t + \phi_{11} X_{t-d} + \dots + \phi_{1L} X_{t-(L-1)d} + \epsilon_{t+s} & Z_t \leq \text{th} \\ \phi_2 + \phi_{20} X_t + \phi_{21} X_{t-d} + \dots + \phi_{2H} X_{t-(H-1)d} + \epsilon_{t+s} & Z_t > \text{th} \end{cases} \quad (3)$$

with Z_t a threshold variable. How is one to define Z_t ? Strictly speaking, in SETAR models Z_t should be one of $\{X_t, X_{t-d}, X_{t-(m-1)d}\}$. We can define the threshold variable Z_t via the *threshold delay* δ , such that

$$Z_t = X_{t-\delta d}$$

Using this formulation, you can specify SETAR models with:

```
R code
```

```
obj <- setar(x, m = , d = , steps = , thDelay = )
```

where `thDelay` stands for the above defined δ , and must be an integer number between 0 and $m - 1$.

For greater flexibility, you can also define the threshold variable as an arbitrary linear combination of lagged time series values:

$$Z_t = \beta_1 X_t + \beta_2 X_{t-1} + \dots + \beta_m X_{t-(m-1)d}$$

In R this is implemented as follows:

```
R code
```

```
obj <- setar(x, m = , d = , steps = , mTh = )
```

where `mTh` stands for β , and takes the form of a vector of real coefficients of length m .

Finally, Z_t can be an external variable. This is obtained with the call:

```
R code
```

```
obj <- setar(x, m = , d = , steps = , thVar = )
```

where `thVar` is the vector containing the threshold variable values.

The threshold variable isn't the only additional parameter governing the SETAR model. One can specify the *low* and *high* regime autoregressive orders L and H . These can be specified with the arguments `mL` and `mH`, respectively:

```
R code
```

```
obj <- setar(x, m = , d = , steps = , thDelay = , mL = , mH = )
```

If not specified, `mL` and `mH` defaults to `m`.

Another hyper-parameter one can specify is the threshold value `th`, via the additional argument `th`. If not specified, this is estimated by fitting the model for a grid of different, reasonable values of `th`, and taking the best fit as the final `th` estimate.

Note that, conditional on $\{Z_t \leq \text{th}\}$, the model is linear. So, for a fixed threshold value, the CLS estimation is straightforward.

The summary command for this model returns asymptotic standard errors for the estimated ϕ coefficients, based on the assumption that ϵ_t are normally distributed.

3.3 LSTAR models

The LSTAR model can be viewed as a generalization of the above defined SETAR model:

$$X_{t+s} = (\phi_1 + \phi_{10}X_t + \phi_{11}X_{t-d} + \dots + \phi_{1L}X_{t-(L-1)d})(1 - G(Z_t, \gamma, \text{th})) \\ + (\phi_2 + \phi_{20}X_t + \phi_{21}X_{t-d} + \dots + \phi_{2H}X_{t-(H-1)d})G(Z_t, \gamma, \text{th}) + \epsilon_{t+s}$$

with G the logistic function, and Z_t the threshold variable. For Z_t , L and H specification, the same convention as that of SETAR models is followed. In addition, for LSTAR models one has to specify some starting values for all the parameters to be estimated: $(\phi, \gamma, \text{th})$.

Estimation is done by analytically determining ϕ_1 and ϕ_2 (through linear regression) and then minimizing residuals sum of squares with respect to th and γ . These two steps are repeated until convergence is achieved.

3.4 Neural Network models

A neural network model with linear output, D *hidden units* and activation function g , is represented as:

$$x_{t+s} = \beta_0 + \sum_{j=1}^D \beta_j g(\gamma_{0j} + \sum_{i=1}^m \gamma_{ij} x_{t-(i-1)d}) \quad (4)$$

For the implementation the **nnet** package is used, so please refer to the **nnet** package documentation for more details.

The only additional argument for specifying this model is the number of hidden units **size**, which stands for the above defined D :

R code

```
obj <- nnetTs(x, m = , d = , steps = , size = )
```

The estimation is done via CLS. No additional summary informations are available for this model.

3.5 Additive Autoregressive models

A non-parametric additive model (a GAM, Generalized Additive Model), of the form:

$$x_{t+s} = \mu + \sum_{i=1}^m s_i(x_{t-(i-1)d}) \quad (5)$$

where s_i are smooth functions represented by penalized cubic regression splines. They are estimated, along with their degree of smoothing, using the **mgcv** package [8].

No additional parameters are required for this model:

R code

```
obj <- aar(x, m = , d = , steps = )
```

Some diagnostic plots and summaries are provided for this model, adapted from those produced by **mgcv**.

3.6 Model selection

A common task in time series modelling is *fine tuning* of the hyper-parameters. R is a complete programming language, so the user can easily define his error criterion, fit a set of models and chose the best between them. However, the `tsDyn` package provides some helpful functions for this task.

For SETAR models, there is the `selectSETAR` function. The time series, the embedding parameters and a vector of values for each provided hyper-parameter is passed to this function. The routine then tries to fit the model for the full grid of hyper-parameter values, and gives as output a list of the best combinations found. So, for example:

```

R code
-----
x <- log10(lynx)
selectSETAR(x, m = 3, mL = 1:3, mH = 1:3, thSteps = 5, thDelay = 0:2)

```

```

output
-----
Searching on 73 possible threshold values within regimes with sufficient ( 15% ) number of observations
Searching on 219 combinations of threshold, thDelay, ML and MM
Results of the grid search for 1 threshold

```

	thDelay	mL	mH	th	pooled-AIC
1	2	1	1	2.940018	-15.612619
2	2	1	1	2.907411	-12.252530
3	2	1	1	3.000000	-10.638946
4	2	1	1	2.894316	-9.038100
5	2	1	1	2.980912	-7.812850
6	2	1	1	2.879669	-6.462867
7	2	1	1	2.828660	-4.661813
8	2	1	1	2.820858	-4.382119
9	2	1	1	2.835056	-3.043105
10	2	1	1	2.866878	-2.127289

tries to fit $3 \times 3 \times 3 \times 5$ models, one for each combination of `mL`, `mH`, `thDelay` and `th`, and returns the best combinations w.r.t. the AIC criterion.

Totally analogous are the `selectLSTAR` and `selectNNET` functions, for which we refer to the online documentation.

4 Case study

We herein analyse the Canadian lynx data set. This consists of annual records of the numbers of the Canadian lynx trapped in the Mackenzie River district of North-west Canada for the period 1821-1934.

The time series, named `lynx`, is available in a default R installation, so one can type directly, in an R session:

R code

```
str(lynx)
```

output

```
Time-Series [1:114] from 1821 to 1934: 269 321 585 871 1475 ...
```

R code

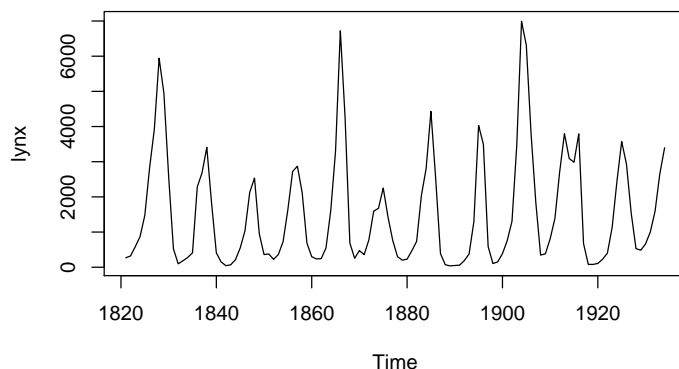
```
summary(lynx)
```

output

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
39.0	348.2	771.0	1538.0	2567.0	6991.0

R code

```
plot(lynx)
```



Here we will roughly follow the analysis in Tong [7].

4.1 Explorative analysis

First, we log transform the data:

R code

```
x <- log10(lynx)
```

Plot of the time series and time-inverted time series:

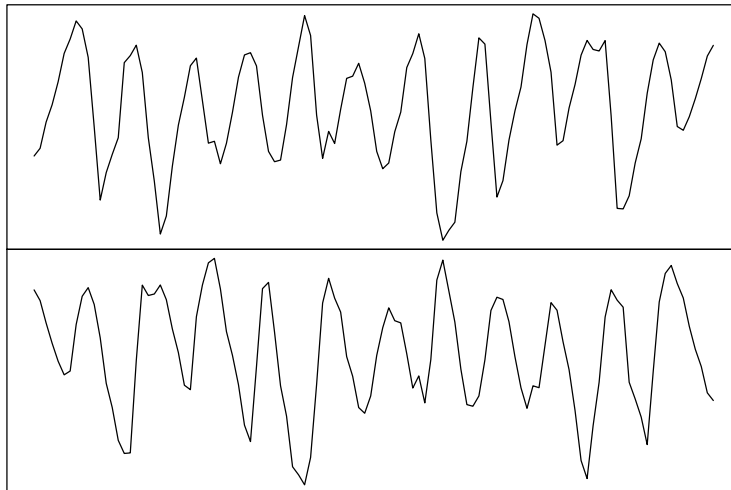
R code

```
par(mfrow = c(2, 1), mar = c(0, 0, 0, 0))
plot(x, ax = F)
```

```

box()
plot(x[length(x):1], type = "l", ax = F)
box()

```

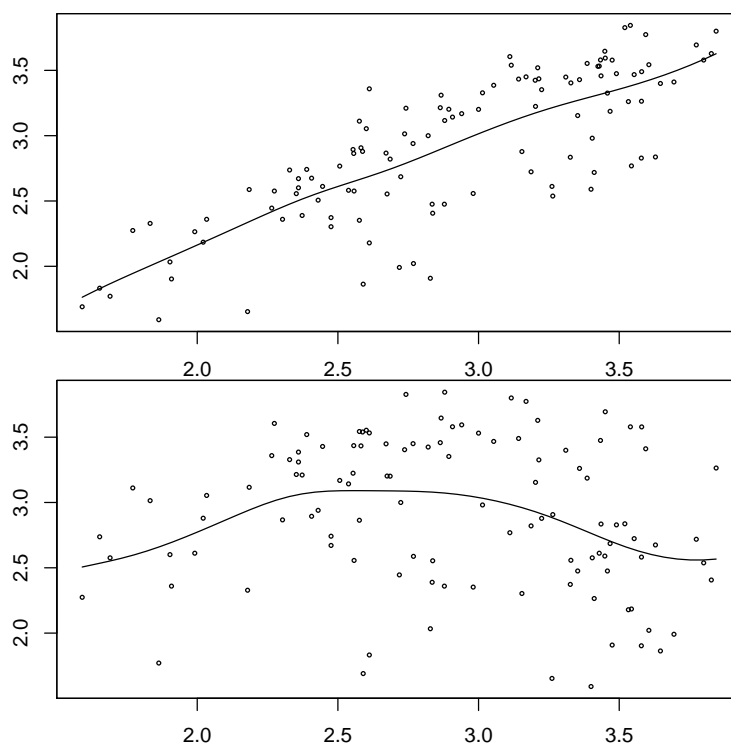


Nonparametric regression function of X_t versus X_{t-1} and of X_t versus X_{t-3} (kernel estimation):

```

R code
par(mfrow = c(2, 1), mar = c(2, 2, 0, 0))
autopairs(x, lag = 1, type = "regression", GUI = FALSE)
autopairs(x, lag = 3, type = "regression", GUI = FALSE)

```

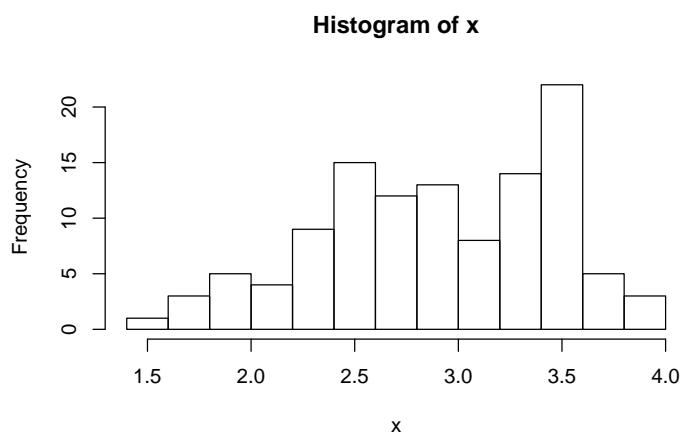


For lag 3 (bottom plot), a linear approximation for the regression function may be questionable.

The marginal histogram of data shows bimodality:

R code

```
hist(x, br = 13)
```



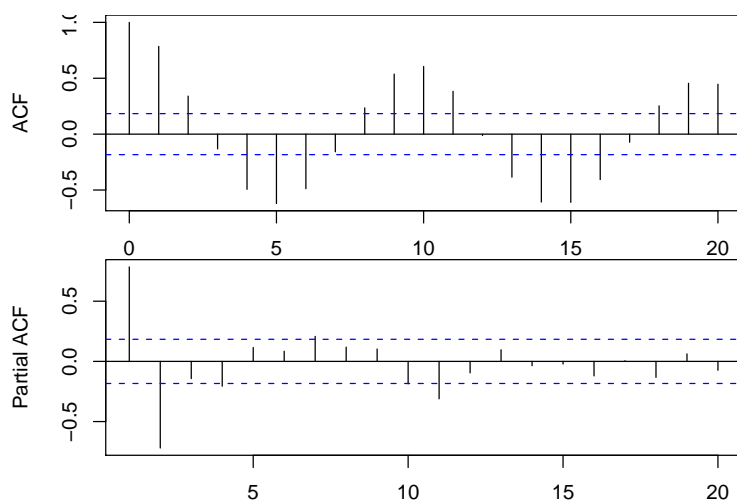
Global and partial autocorrelation:

R code

```

par(mfrow = c(2, 1), mar = c(2, 4, 0, 0))
acf(x)
pacf(x)

```

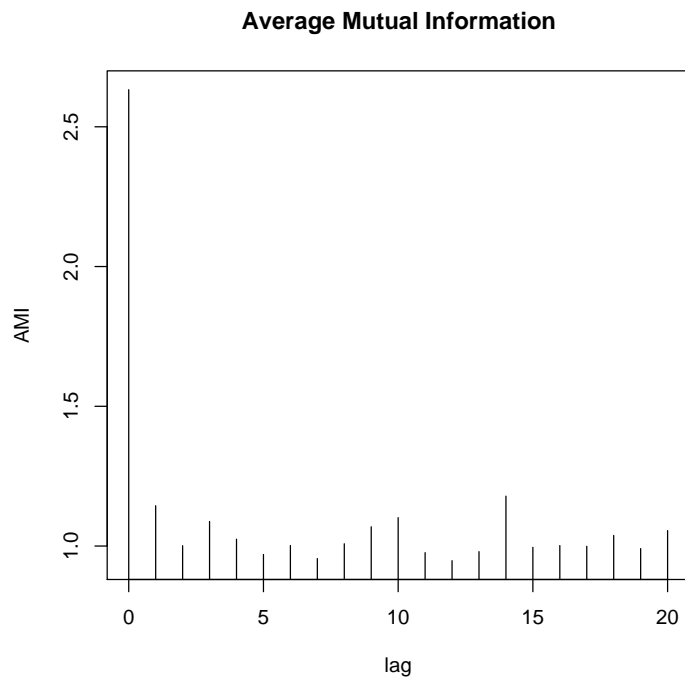


The `tseriesChaos` package offers some other explorative tools tipycal of nonlinear time series analysis. The Average Mutual Information (see online help for further explanation):

```

library(tseriesChaos)
mutual(x)

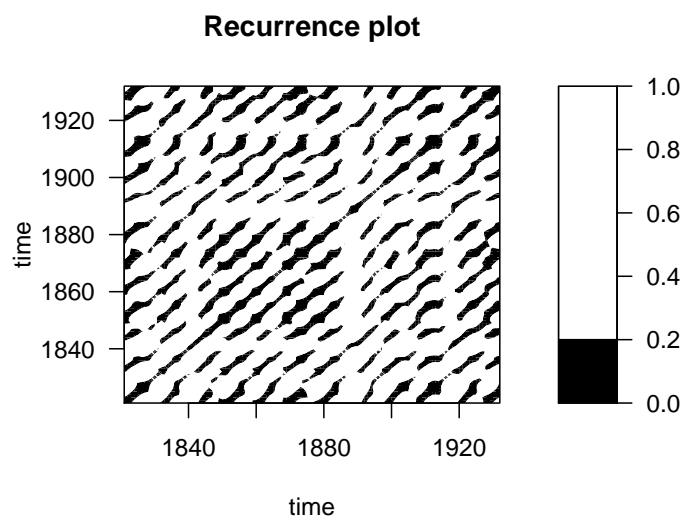
```



Recurrence plot (see online help):

R code

```
recurr(x, m = 3, d = 1, levels = c(0, 0.2, 1))
```

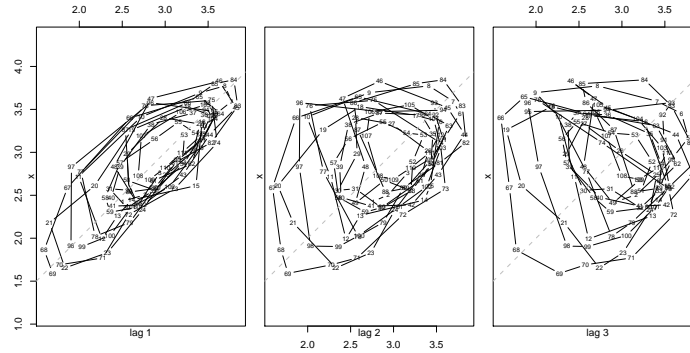


From this plot, deterministic cycles appears from the embedding-reconstructed underlying dynamics.

Directed lines are a tipycal tool for time series explorations. The `lag.plot` function in the base `stats` package does this well:

R code

```
lag.plot(x, lags = 3, layout = c(1, 3))
```



Especially for lag 2, a cycle is again evident. Moreover, the void space in the middle is a typical argument for rejecting the bivariate normality of (X_t, X_{t-l}) .

What follows is the application of still-experimental code for testing the conditional mutual independence and linearity for lags 2 and 3:

R code

```
delta.test(x)
```

output

```
eps
m  0.2792 0.5584 0.8376 1.1168
 2   0.02   0.02   0.04   0.02
 3   0.26   0.02   0.02   0.02
```

R code

```
delta.lin.test(x)
```

output

```
eps
m  0.2792 0.5584 0.8376 1.1168
 2   0.28   0.16   0.56   0.56
 3   0.22   0.40   0.16   0.50
```

P-values are reported, labelled with their embedding dimensions m and *window* values ϵ . We reject conditional independence quite easily. There is some trouble instead for deciding to reject or not linearity. See Manzan [5] for a detailed discussion on these tests.

4.2 Model selection

The first model proposed in the literature for these data was an AR(2):

$$X_t = 1.05 + 1.41X_{t-1} - 0.77X_{t-2} + \epsilon_t$$

with $v(\epsilon_t) = \sigma^2 = 0.04591$.

This can be estimated with `tsDyn` using the command:

```

R code
mod.ar <- linear(x, m = 2)
mod.ar

```

```

output
Non linear autoregressive model

AR model
Coefficients:
      const      phi.1      phi.2
1.0576005  1.3842377 -0.7477757

```

As an improvement to the AR model, we may consider applying a SETAR(2; 2,2) model with threshold delay $\delta = 1$. In R:

```

R code
mod.setar <- setar(x, m = 2, mL = 2, mH = 2, thDelay = 1)
mod.setar

```

```

output
Non linear autoregressive model

SETAR model ( 2 regimes)
Coefficients:
Low regime:
      phiL.1      phiL.2      const L
1.2642793 -0.4284292  0.5884369

High regime:
      phiH.1      phiH.2      const H
1.599254 -1.011575  1.165692

Threshold:
-Variable: Z(t) = + (0) X(t)+ (1)X(t-1)
-Value: 3.31
Proportion of points in low regime: 69.64%      High regime: 30.36%

```

So, the fitted model may be written as:

$$X_{t+1} = \begin{cases} 0.588 + 1.264X_t - 0.428X_{t-1} & X_{t-1} \leq 3.31 \\ 1.166 + 1.599X_t - 1.012X_{t-1} & X_{t-1} > 3.31 \end{cases}$$

For an automatic comparison, we may fit different linear and nonlinear models and directly compare some measures of their fit:

```

R code
mod <- list()
mod[["linear"]] <- linear(x, m = 2)
mod[["setar"]] <- setar(x, m = 2, thDelay = 1)
mod[["lstar"]] <- lstar(x, m = 2, thDelay = 1)

```

```

output
Using maximum autoregressive order for low regime: mL = 2
Using maximum autoregressive order for high regime: mH = 2
Performing grid search for starting values...
Starting values fixed: gamma = 11 , th = 3.338679 ; SSE = 4.337657
Optimization algorithm converged
Optimized values fixed for regime 2 : gamma = 11.00002 , th = 3.340093

```

```

R code
mod[["nnetTs"]] <- nnetTs(x, m = 2, size = 3)
mod[["aar"]] <- aar(x, m = 2)

```

Now the `mod` object contains a labelled list of fitted `nlar` models. As an example, we can compare them in term of the AIC and MAPE index:

```

R code
sapply(mod, AIC)

```

```

output
linear      setar      lstar      nnetTs      aar
-333.8737 -358.3740 -356.6509 -344.4731 -328.0813

```

```

R code
sapply(mod, MAPE)

```

```

output
linear      setar      lstar      nnetTs      aar
0.06801955 0.05648596 0.05580458 0.05709281 0.05951108

```

From this comparison, the SETAR model seems to be the best.

More detailed diagnostics can be extracted:

R code

```
summary(mod[["setar"]])
```

output

Non linear autoregressive model

SETAR model (2 regimes)

Coefficients:

Low regime:

phiL.1	phiL.2	const L
1.2642793	-0.4284292	0.5884369

High regime:

phiH.1	phiH.2	const H
1.599254	-1.011575	1.165692

Threshold:

-Variable: $Z(t) = + (0) X(t) + (1)X(t-1)$

-Value: 3.31

Proportion of points in low regime: 69.64% High regime: 30.36%

Residuals:

Min	1Q	Median	3Q	Max
-0.571121	-0.109431	0.017641	0.116468	0.516270

Fit:

residuals variance = 0.03814, AIC = -358, MAPE = 5.649%

Coefficient(s):

	Estimate	Std. Error	t value	Pr(> t)
const L	0.588437	0.143307	4.1061	7.844e-05 ***
phiL.1	1.264279	0.065256	19.3741	< 2.2e-16 ***
phiL.2	-0.428429	0.077487	-5.5291	2.260e-07 ***
const H	1.165692	0.876606	1.3298	0.1863928
phiH.1	1.599254	0.108966	14.6767	< 2.2e-16 ***
phiH.2	-1.011575	0.265011	-3.8171	0.0002255 ***

Signif. codes: 0

Threshold

Variable: $Z(t) = + (0) X(t) + (1) X(t-1)$

Value: 3.31

More diagnostic plots can be displayed using the command:

R code

```
plot(mod[["setar"]])
```

4.3 Out-of-sample forecasting

Fit models on first 104 observations:

R code

```
set.seed(10)
mod.test <- list()
x.train <- window(x, end = 1924)
x.test <- window(x, start = 1925)
mod.test[["linear"]] <- linear(x.train, m = 2)
mod.test[["setar"]] <- setar(x.train, m = 2, thDelay = 1)
mod.test[["lstar"]] <- lstar(x.train, m = 2, thDelay = 1, trace = FALSE,
+   control = list(maxit = 1e+05))
mod.test[["nnet"]] <- nnetTs(x.train, m = 2, size = 3, control = list(maxit = 1e+05))
```

output

```
# weights: 13
initial value 1009.386247
iter 10 value 5.610020
iter 20 value 5.422071
iter 30 value 5.408916
iter 40 value 5.319344
iter 50 value 5.080814
iter 60 value 4.862847
iter 70 value 4.596344
iter 80 value 4.436638
iter 90 value 4.296703
iter 100 value 4.266607
iter 110 value 4.248017
iter 120 value 4.232015
iter 130 value 4.218466
iter 140 value 4.215550
final value 4.215539
converged
```

```

R code
mod.test[["aar"]] <- aar(x.train, m = 2)

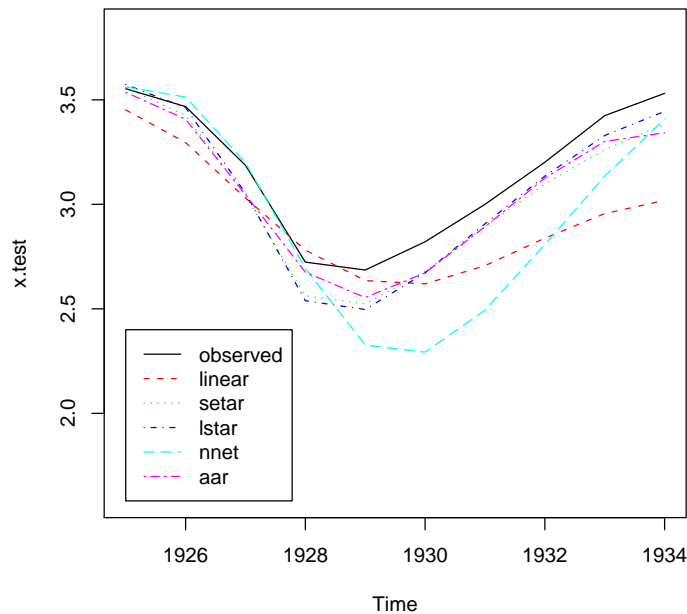
```

Compare forecasts with real last 10 observed values:

```

R code
frc.test <- lapply(mod.test, predict, n.ahead = 10)
plot(x.test, ylim = range(x))
for (i in 1:length(frc.test)) lines(frc.test[[i]], lty = i +
+   1, col = i + 1)
legend(1925, 2.4, lty = 1:(length(frc.test) + 1), col = 1:(length(frc.test) +
+   1), legend = c("observed", names(frc.test)))

```



From this visual comparison, the SETAR(2; 2,2) model seems to be one of the bests.

4.4 Inspecting model skeleton

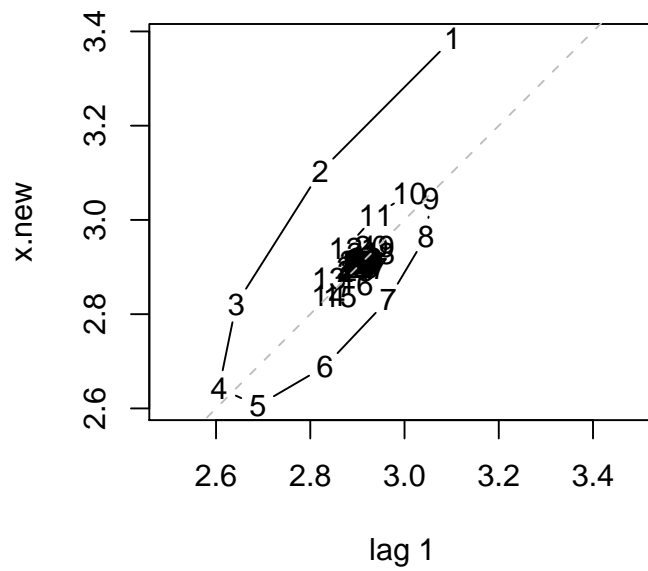
An interesting task can be inspecting the fitted model skeleton.

This can be achieved by comparing the forecasting results under each model.

```

R code
x.new <- predict(mod[["linear"]], n.ahead = 100)
lag.plot(x.new, 1)

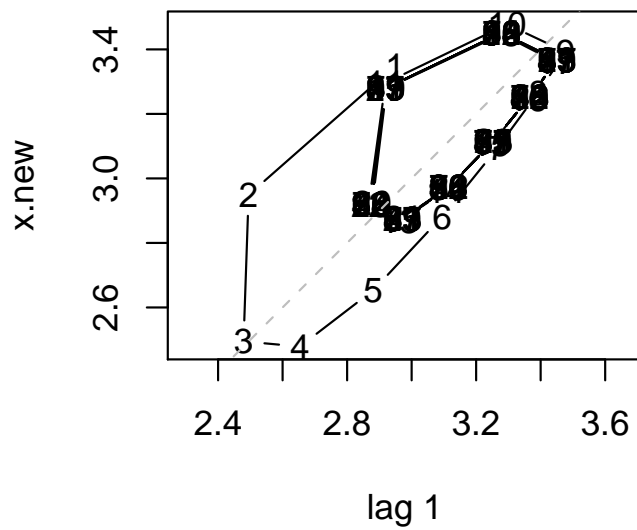
```



A fixed point, i.e. the only possible stationary solution with a linear model.

R code

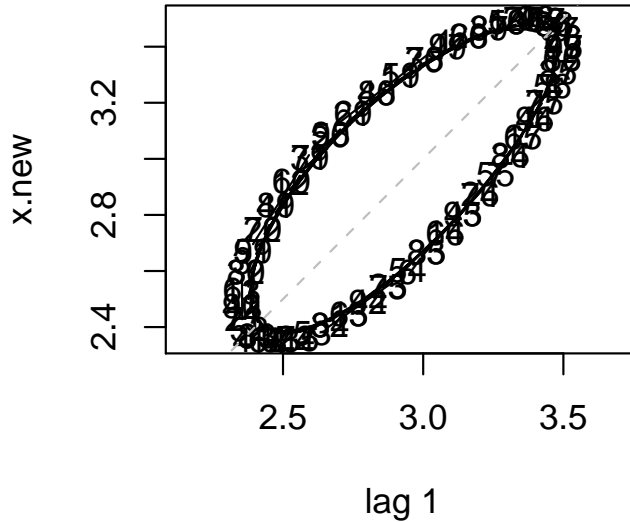
```
x.new <- predict(mod[["setar"]], n.ahead = 100)
lag.plot(x.new, 1)
```



A stable periodic cycle.

R code

```
x.new <- predict(mod[["nnetTs"]], n.ahead = 100)
lag.plot(x.new, 1)
```



Appears to be a quasiperiodic cycle lying on an invariant curve.

5 Sensitivity on initial conditions

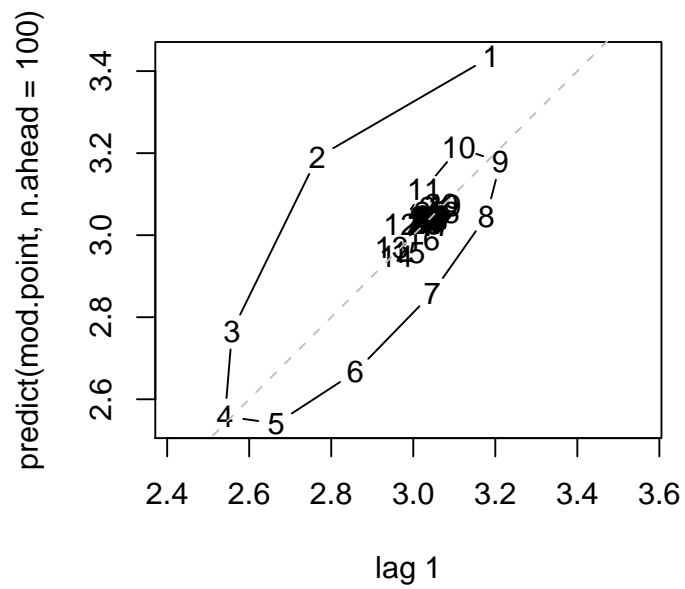
In the previous section we observed skeletons with cyclical or limit fixed point behaviour.

Neural networks and SETAR models can explain also different types of attractors. For this data set, Tong [7] showed that particular types of SETAR models can yield to fixed limit points as well as unstable orbits and *possibly chaotic* systems.

For example, a fixed limit point:

```
R code
```

```
mod.point <- setar(x, m = 10, mL = 3, mH = 10, thDelay = 0, th = 3.12)
lag.plot(predict(mod.point, n.ahead = 100))
```

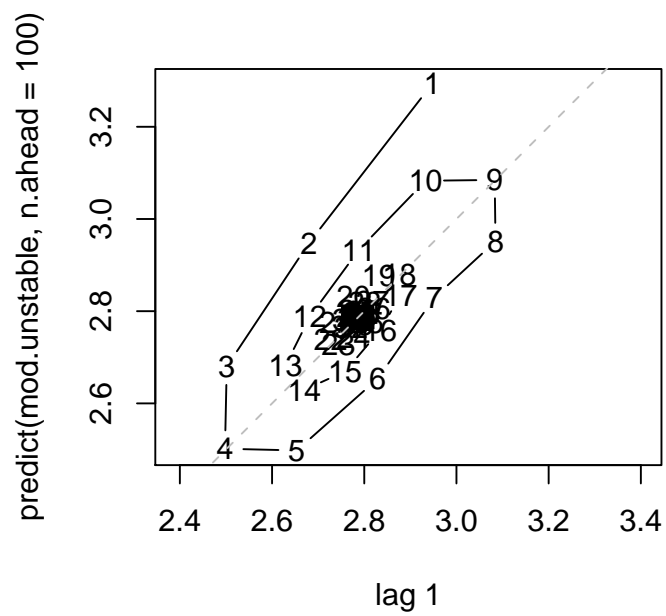


Unstable orbit:

```

R code
mod.unstable <- setar(x, m = 9, mL = 9, mH = 6, thDelay = 4,
+   th = 2.61)
lag.plot(predict(mod.unstable, n.ahead = 100))

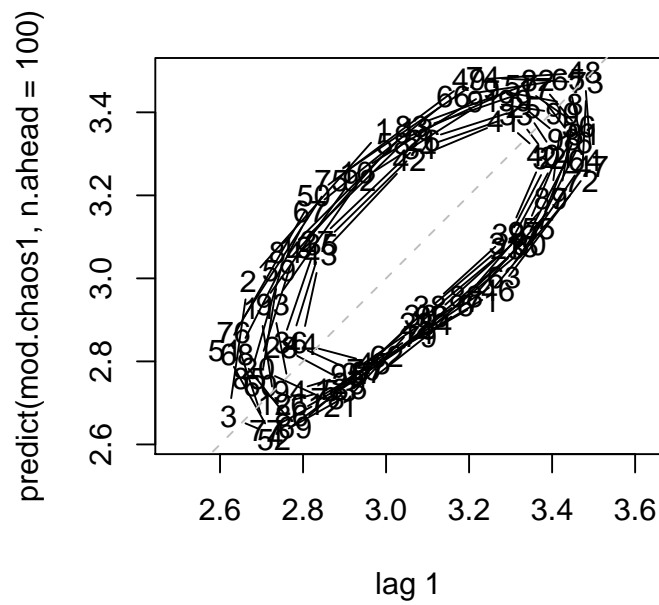
```



Possibly chaotic systems:

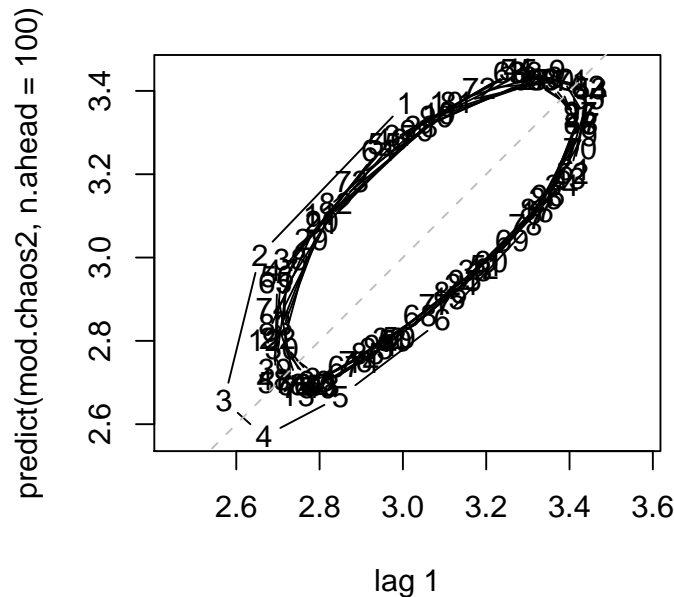
R code

```
mod.chaos1 <- setar(x, m = 5, mL = 5, mH = 3, thDelay = 1, th = 2.78)
lag.plot(predict(mod.chaos1, n.ahead = 100))
```



R code

```
mod.chaos2 <- setar(x, m = 5, mL = 5, mH = 3, thDelay = 1, th = 2.95)
lag.plot(predict(mod.chaos2, n.ahead = 100))
```



For a given fitted model, we can try estimating the maximal Lyapunov exponent with the Kantz algorithm using the `lyap_k` function in the `tseriesChaos` package [6, 4]. This function takes as input an observed time series, so we can proceed as follows:

1. generate N observations from the model
2. add a little observational noise (otherwise the Kantz algorithm will fail)
3. apply the `lyap_k` function to the generated time series

Follows the R code for analysing the selected SETAR(2; 2,2) model of the previous paragraph and the possibly chaotic SETAR(2; 5,3) just seen above.

R code

```

N <- 1000
x.new <- predict(mod[["setar"]], n.ahead = N)
x.new <- x.new + rnorm(N, sd = sd(x.new)/100)
ly <- lyap_k(x.new, m = 2, d = 1, t = 1, k = 2, ref = 750, s = 200,
+   eps = sd(x.new)/10)

```

output

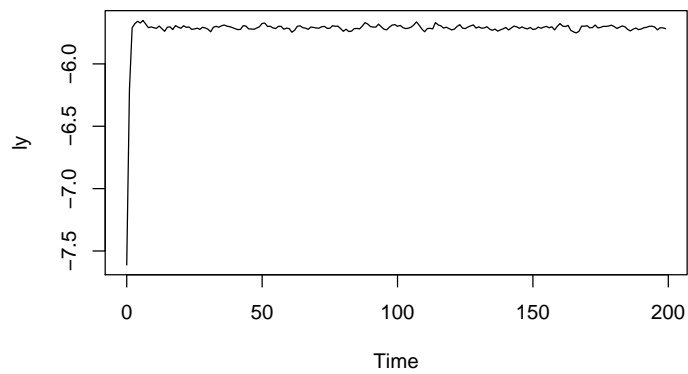
```

Finding nearests
Keeping 741 reference points
Following points

```

R code

```
plot(ly)
```



There is no scaling region, so the maximal Lyapunov exponent can be assumed to be ≤ 0 .

R code

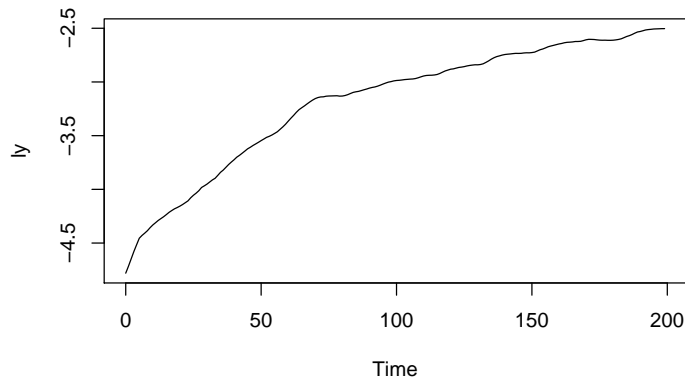
```
x.new <- predict(mod.chaos2, n.ahead = N)
x.new <- x.new + rnorm(N, sd = sd(x.new)/100)
ly <- lyap_k(x.new, m = 5, d = 1, t = 1, k = 2, ref = 750, s = 200,
+   eps = sd(x.new)/10)
```

output

```
Finding nearests
Keeping 732 reference points
Following points
```

R code

```
plot(ly)
```



Here there is a scaling region. The final λ estimate for this time series is the slope of the plotted curve in that region:

R code	
<code>lyap(ly, start = 6, end = 70)</code>	
output	
(Intercept)	lambda
-4.53873652	0.01979557

At this point a natural question can be: *why not use directly the original time series as input to `lyap_k` instead of model-generated observations?* The answer here is that we have a too short time series for successfully applying the Kantz algorithm, so a preliminary modelling for generating more observations is necessary.

References

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- [5] Sebastiano Manzan, *Essays in nonlinear economic dynamics*, Thela Thesis, 2003.
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